Advanced Statistical Topics

Day 2 - Penalized regression

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Today's topics

- Penalized regression
- Revisiting Cross validation
- Variants and extensions

Handle complicated modeling aspects with many potential predictors.

Multiple regression

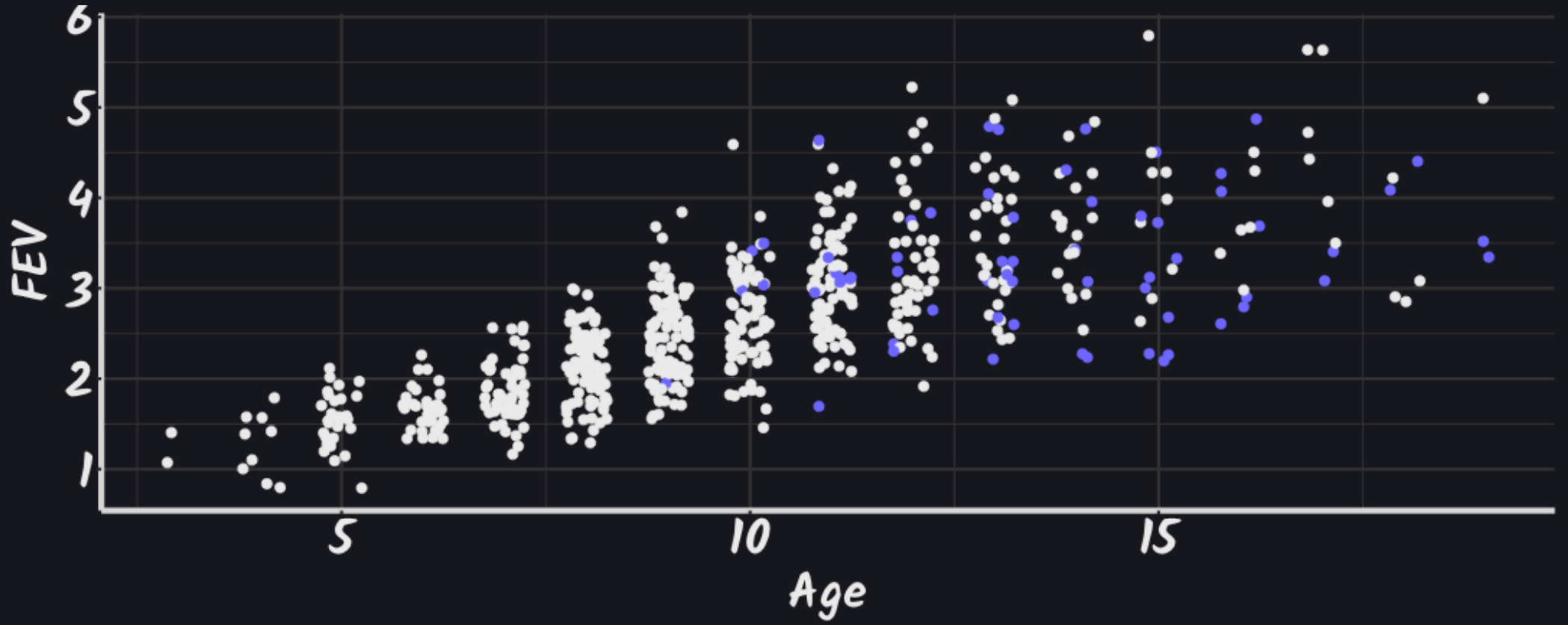
For the i'th individual,

$$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \cdots + \beta_p x_{pi} + \epsilon_i$$

- Each coefficient, β_j, expresses the expected change in y when x_j changes one unit and the remaining explanatory variables are fixed.
- β_j measures the effect of x_j on the response taking the effect of the other variables into account.
- Might have a significant effect of x_1 in the univariate model, while its effect in the mult. lin. model is non-significant. Its effect is explained by the other explanatory variables.

Example - FEV

654 children. Does smoking influence lung capacity?



Example - FEV

```
lm(FEV ~ Smoke + Age + Ht + Gender, data=fev) |> broom::tidy()
```

```
# A tibble: 5 × 5
  term        estimate std.error statistic  p.value
  <chr>          <dbl>     <dbl>     <dbl>    <dbl>
1 (Intercept)  -4.45     0.224      -19.9  3.53e-69
2 SmokeYes     -0.0898   0.0594      -1.51 1.31e- 1
3 Age           0.0661   0.00950      6.95 8.69e-12
4 Ht            4.10     0.188       21.8  1.89e-79
5 GenderBoy     0.156    0.0333       4.68 3.46e- 6
```

Generalized linear models

Expand the multiple regression model by applying a transformation with a suitable function. This gives a so-called *generalized* linear model

$$g(\mathbb{E}[Y_i|X_{1i}, \ldots, X_{pi}]) = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_p X_{pi}$$

or equivalently

$$\mathbb{E}[Y_i|X_{1i}, \ldots, X_{pi}] = g^{-1}(\beta_0 + \beta_1 X_{1i} + \cdots + \beta_p X_{pi})$$

where the function g in principle is any function that maps the population mean into the linear predictor. g is called the *link function*.

Generalized linear models

Equivalently

$$\mathbb{E}[Y_i|X_{1i}, \ldots, X_{pi}] = g^{-1}(X\beta)$$

Only going to consider Gaussian data with identity link:

$$\mathbb{E}[Y_i|X_{1i}, \ldots, X_{pi}] = X\beta$$

and

$$Y = X\beta + \epsilon$$

OLS estimation

From first year statistics the ordinary least squares estimator is

$$\hat{\beta} = (X^t X)^{-1} X^t y$$

Written as least squares so $\hat{\beta}$ minimizes

$$(y - X\hat{\beta})^t (y - X\hat{\beta})$$

or "just"

$$\sum_i (y_i - X_i \beta)^2$$

Penalized regression

Let $Y = (Y_1, \ldots, Y_n)$ be vector of outcomes in $\mathbb{R}$, and
$X = (X_1, \ldots, X_n)$ a set of M predictors for each observation.

$$Y = X\beta + \epsilon$$

The Lasso estimates β by minimizing the penalized LS function

$$Z_n(\beta) = \frac{1}{n}(Y - X\beta)^t(Y - X\beta) + \lambda_n \|\beta\|$$

so

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^M} Z_n(\beta)$$

Properties of regularized regression

Even for $M > n$ or $M \gg n$ regularized regression methods can:

- select a sparse model
- lead to accurate prediction

Limitations of LASSO type regularization:

- not consistent in variable selection
- non-standard limiting distribution
- no oracle property
- multiple testing problem

Example: depression

- Outcome: depression (scale 0-40).
- Input: 384 normalized genes expression value for 40 persons

```
      y    x1    x2    x3    x4    x5    x6
[1,] 22 -0.63 -0.16 -0.57 -0.51  0.43  0.41
[2,] 15  0.18 -0.25 -0.14  1.34 -0.24  1.69
[3,] 21 -0.84  0.70  1.18 -0.21  1.06  1.59
[4,] 27  1.60  0.56 -1.52 -0.18  0.89 -0.33
[5,] 12  0.33 -0.69  0.59 -0.10 -0.62 -2.29
```

Example - 384 linear regressions

```
pvals <- pt(
            -abs(MESS::mfastLmCpp(y, x)$tstat)
            , df=38)

head(sort(pvals))
```

```
[1] 0.0003440182 0.0007709503 0.0013871463 0.0014807050
[5] 0.0018858497 0.0026072106
```

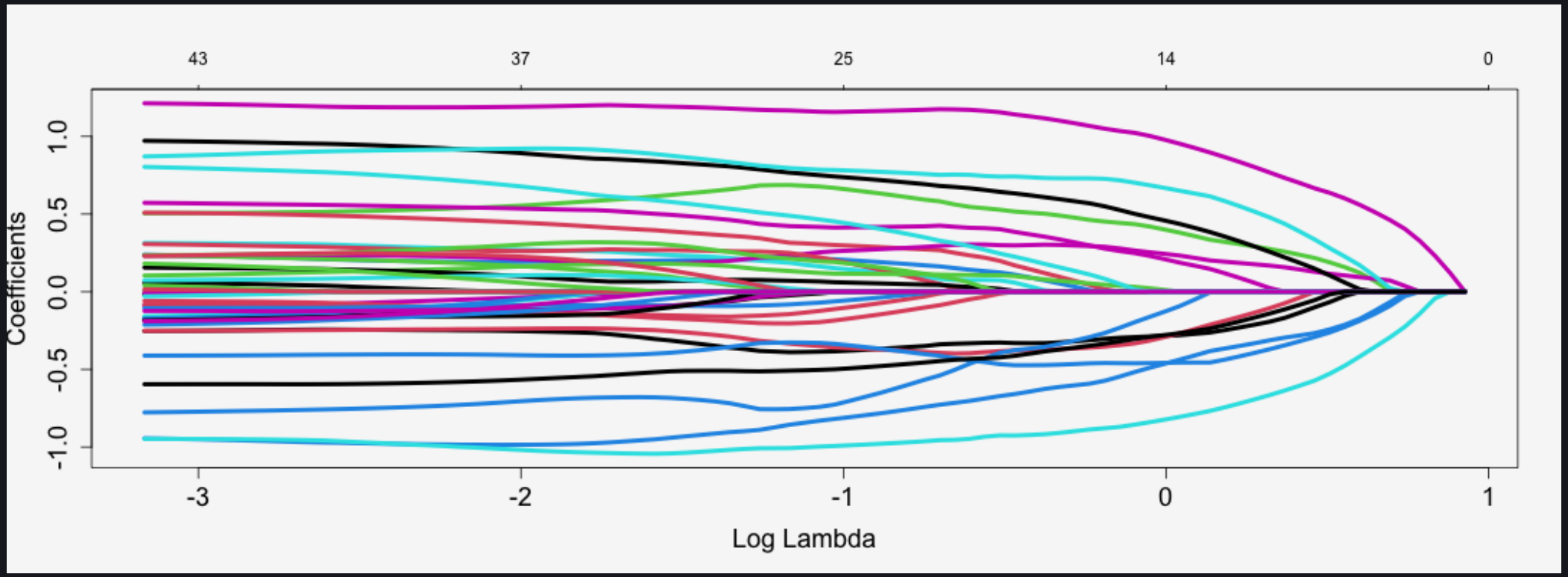
Bonferroni, Holm and FDR "kills" all but variable 264. Not signifikant.

```
384*head(sort(pvals))
```

```
[1] 0.1321030 0.2960449 0.5326642 0.5685907 0.7241663
[6] 1.0011689
```

Lasso

```
library("glmnet")
res <- glmnet(x, y) ; plot(res, xvar="lambda", lwd=4, cex.axis=1.5, ce
```



Results

```
lambda <- exp(0.5) ; coef(res, s=lambda)
```

```
385 x 1 sparse Matrix of class "dgCMatrix"
                        s1
(Intercept)  1.687416e+01
V1           .
V2           .
V3           .
V4           .
V5           .
V6           .
V7           .
V8           .
V9           .
V10          .
V11          .
V12
```

```
nonzero <- which(coef(res, s=lambda) != 0) ; nonzero
```

```
 [1]   1  37  40  43 112 116 146 194 229 265 297 381
```

```
cbind(nonzero, coef(res, s=lambda)[nonzero])
```

```
      nonzero
 [1,]       1  1.687416e+01
 [2,]      37 -7.199668e-02
 [3,]      40 -3.854751e-05
 [4,]      43  1.933381e-01
 [5,]     112 -2.462418e-01
 [6,]     116 -5.349191e-01
 [7,]     146  1.172788e-01
 [8,]     194  1.117310e-01
 [9,]     229  2.889143e-01
[10,]     265  6.411096e-01
[11,]     297 -2.668680e-01
[12,]     381 -2.295539e-02
```

Evaluating regularized regression results

How to evalute the results from regularized regression?

- We get a list of parameters: $\hat{\beta}_{(1)}, \hat{\beta}_{(2)}, \hat{\beta}_{(3)}, \ldots, 0, 0, \ldots$
- They are all shrunk and biased towards 0.
- How do we know which of them are significant?

Classical approach: Test hypothesis that the jth predictor is significant

$$H_0 : \beta_j = 0$$

Potential problems with multiple testing, selection algorithm, bias, lack of small-sample test statistic, ...

Debiasing

```
selected <- nonzero[-1]-1 ; selected
```

```
 [1]  36  39  42 111 115 145 193 228 264 296 380
```

```
broom::tidy(lm(y ~ x[,selected]))
```

```
# A tibble: 12 × 5
   term             estimate std.error statistic  p.value
   <chr>               <dbl>     <dbl>     <dbl>    <dbl>
 1 (Intercept)        16.9       0.343     49.3  9.16e-29
 2 x[, selected]1     -0.874     0.353     -2.47 1.98e- 2
 3 x[, selected]2     -0.830     0.300     -2.77 9.94e- 3
 4 x[, selected]3      0.674     0.402      1.68 1.05e- 1
 5 x[, selected]4     -0.595     0.435     -1.37 1.82e- 1
 6 x[, selected]5     -1.29      0.362     -3.57 1.30e- 3
 7 x[, selected]6      0.455     0.400      1.14 2.66e- 1
```

Selective Inference

Compute p-values and CIs for the lasso estimate, at a fixed value of the tuning parameter λ.

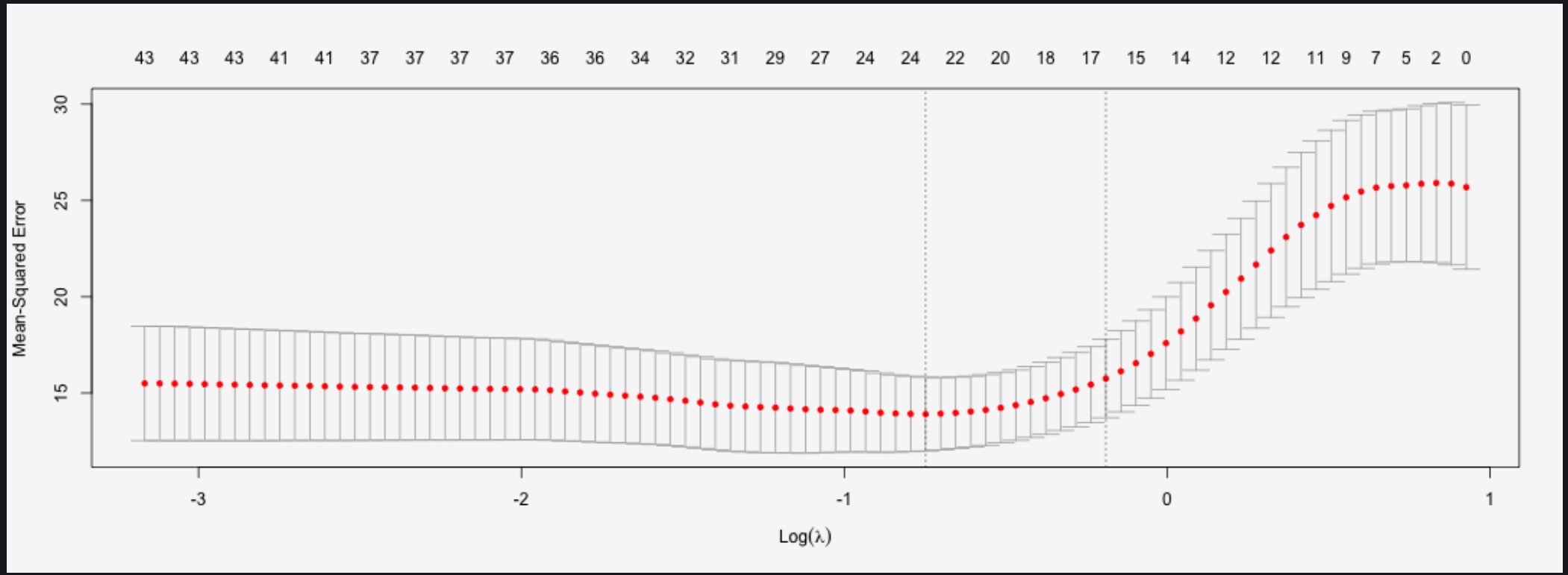
```
library("selectiveInference")
lambda <- 2
n <- 40
beta <- coef(res, s=lambda/n,
             exact=TRUE,x=x,y=y)[-1]
fixedLassoInf(x, y, beta, lambda, sigma=sigma)
```

Caution: quite persnickety

Exercises!

How to choose the penalty?

```
res <- cv.glmnet(x, y)  # 10-fold CV
plot(res, lwd=2)
```



```
res
```

```
Call:  cv.glmnet(x = x, y = y)

Measure: Mean-Squared Error

    Lambda Index Measure    SE Nonzero
min 0.4733    37   13.91 1.913      24
1se 0.8270    25   15.75 2.034      16
```

Penalized regression - Ridge regression

How about using another penalty function:

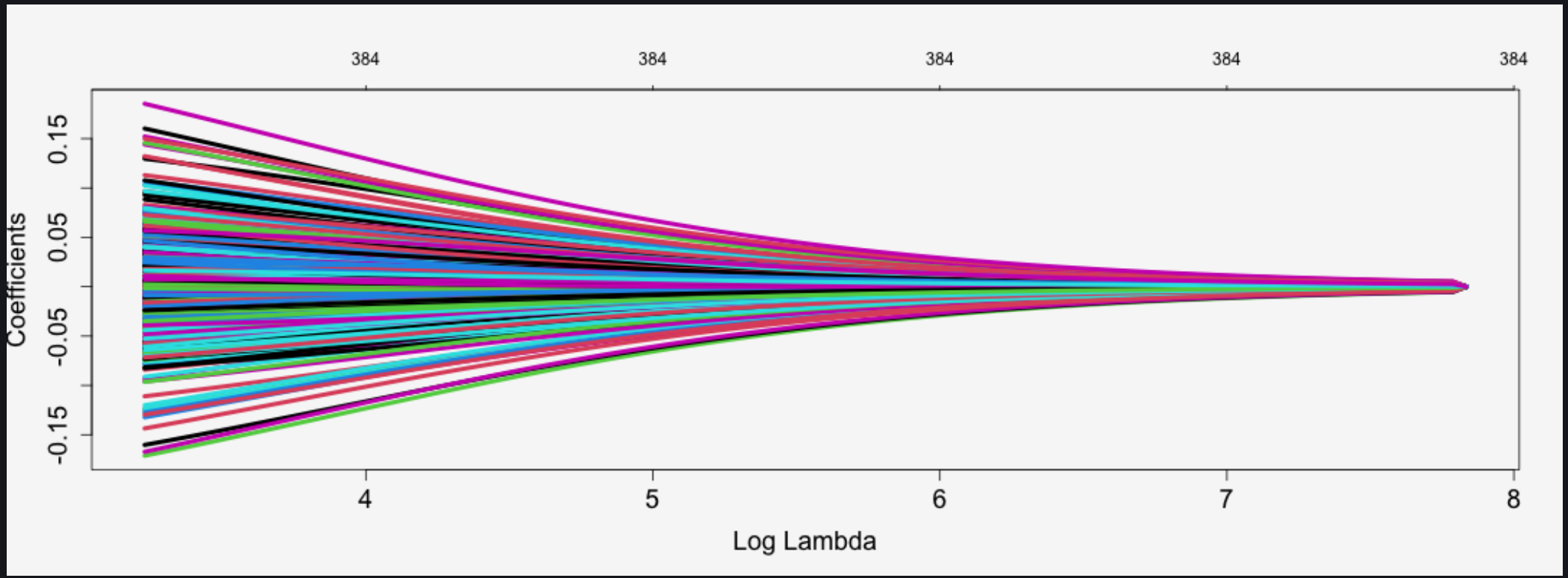
The *Ridge regression* estimates β by minimizing the penalized LS function

$$Z_n(\beta) = \frac{1}{n}(Y - X\beta)^t(Y - X\beta) + \xi_n \|\beta\|^2$$

Ridge regression proceeds by adding a small value, to the diagonal elements of the correlation matrix of the parameters.

Introduces bias but reduces variation.

```
res <- glmnet(x, y, alpha=0)
plot(res, xvar="lambda", lwd=4, cex.axis=1.5, cex.lab=1.5)
```



Penalized regression - elastic net

How about using yet another penalty function:

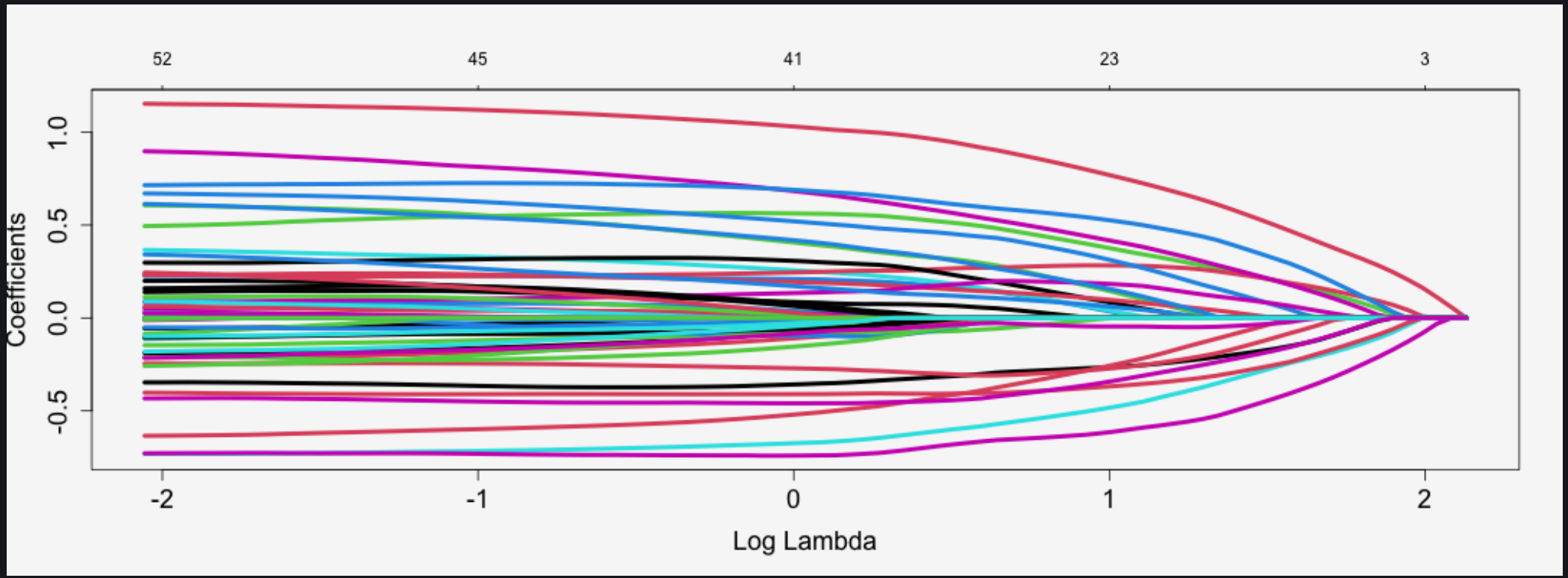
The elastic net regression estimates β by minimizing the penalized LS function

$$Z_n(\beta) = \frac{1}{n}(Y - X\beta)^t(Y - X\beta) + \lambda_n\|\beta\| + \xi_n\|\beta\|^2$$

has Lasso and ridge regression as special cases.

Estimated using an iterative two-step procedure.

```
res <- glmnet(x, y, alpha=.3)
plot(res, xvar="lambda", lwd=4, cex.axis=1.5, cex.lab=1.5)
```



Exercises!

The adaptive lasso

The *adaptive lasso* uses weighted coefficient penalties to obtain better recoverage of the model (the oracle property)

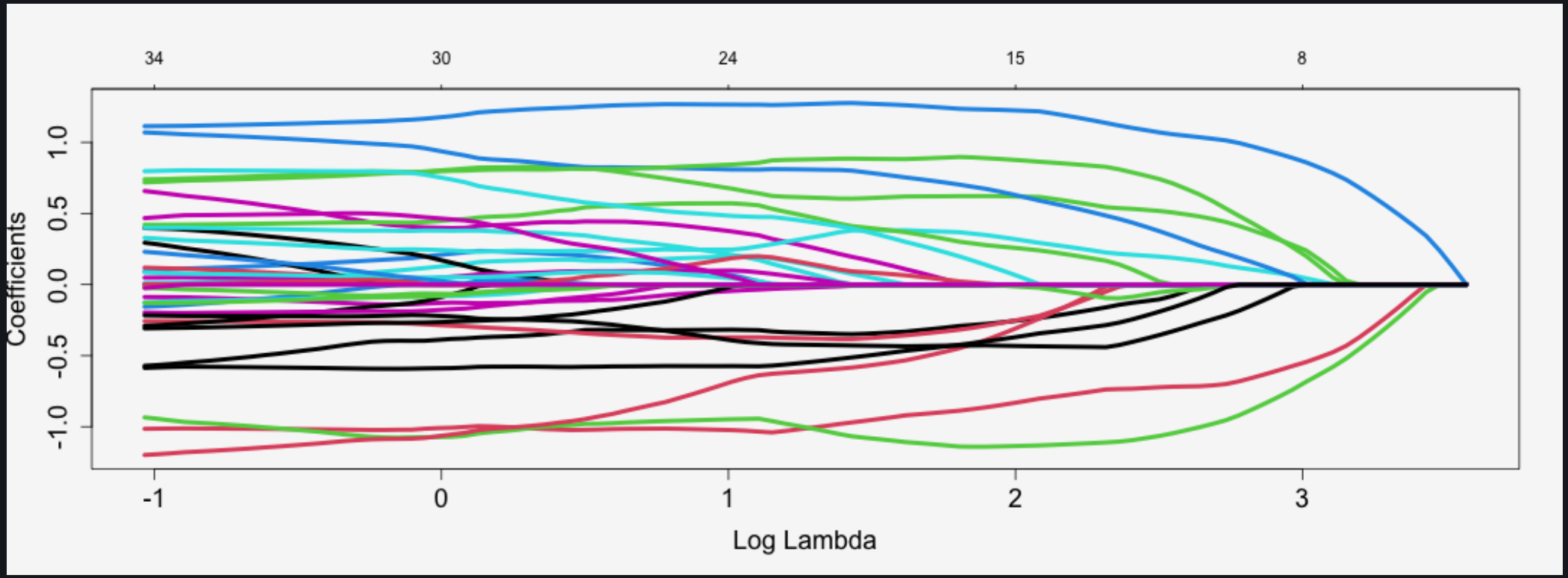
$$Z_n(\beta) = \frac{1}{n}(Y - X\beta)^t(Y - X\beta) + \lambda_n \|\omega\beta\|$$

with $\omega_j = 1/|\hat{\beta}_j|$.

Can use

- LS estimates when $p < N$
- Univariate estimates when $p \gg N$

```
univariate_coefs <- coef(MESS::mfastLmCpp(y, x))
res <- glmnet(x, y, penalty.factor = 1/abs(univariate_coefs))
plot(res, xvar="lambda", lwd=4, cex.axis=1.5, cex.lab=1.5)
```



Group lasso

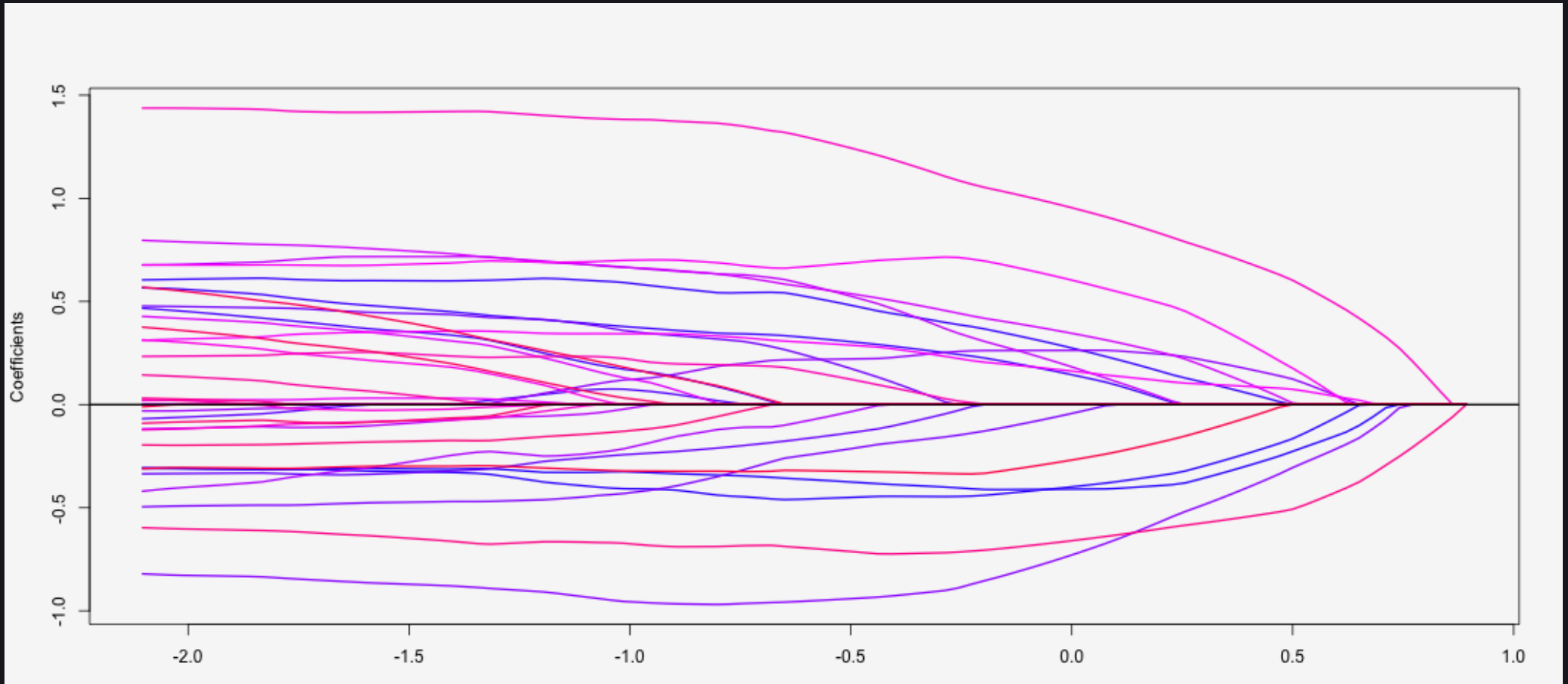
The *group lasso* considers blocks of parameters and places the penalty on each block - either all parameters are non-zero or all zero.

$$Z_n(\beta) = \frac{1}{n}(Y - X\beta)^t(Y - X\beta) + \lambda_n \sum_j \|\theta_j\|$$

where θ_j represents a group of p_j regression coefficients.

Use `grpreg` package with a factor to represent groups (of columns)

```r
library("gglasso")
# Individual groups - sequential
v.group <- c(rep(1, 10), rep(2, 8), rep(3, 6), seq_len(360) + 3)
res <- gglasso(x, y, group = v.group, loss="ls") ; plot(res)
```



Exercises!

Adaptive lasso

Lasso *hates* paying the penalty. Collinearity typically results in only one "representative" *Adaptive lasso* addresses this (and the lacking oracle property) by considering

$$Z_n(\beta) = \|(Y - X\beta)\|_2^2 + \lambda_N \sum_{j=1}^{P} \widehat{\omega}_j |\beta|$$

with

$$\widehat{\omega}_j = \frac{1}{|\widehat{\beta_j}^{\text{OLS}}|}$$

Adaptive lasso

```
library("MESS")
weights <- adaptive.weights(x, y, weight.method = "univariate")
```

```
Warning in adaptive.weights(x, y, weight.method =
"univariate"): using univariate weight method since p>n
```

```
res <- glmnet(x, y, penalty.factor = weights$weights)
```

Highly adaptive lasso

Until now: all our predictors have had a linear relationship to the outcome (by choice).

Idea: any function can be approximated (from data) by a linear combination of spline basis functions.

If we can remove the linearity assumption we can get more flexible (so better) predictions.

Can be used as a smoother - even in low dimensions

Highly adaptive lasso

```
library(hal9001)
hal_fit <- fit_hal(x, y)
preds_hal <- predict(object = hal_fit, new_data = genes)
# MSPE
```

Arguments:

- `max_degree` determines the order of interactions
- `smoothness_orders` - order of smoothness of the regression function. 1 = piecewise linear
- `num_knots` Number of knots for the spline basis functions